Proofs of Propositions

We provide below the proofs of our propositions. \( \overline{p} \) denotes the reverse of path \( p \). \( p^r_s \) denotes a path from node \( s \) to a set of nodes \( V \). We begin by proving a lemma that is needed to prove our propositions.

**Lemma 1** Let \( v, v' \) and \( s \) be nodes in a ground hypergraph whose nodes are all reachable from \( s \). If \( \text{Sym}_s(v,v') \), then \( v \) and \( v' \) have the same number of \( r \)-hyperedges connected to them.

**Proof.** Suppose for a contradiction that \( v \) and \( v' \) respectively have \( n \) and \( n' \) \( r \)-hyperedges connected to them, and \( n > n' \). Let \( p^n_s \) be a path from \( s \) to \( v \), \( r_1, \ldots, r_n \) be the \( r \)-hyperedges that are connected to \( v \), and \( V_1, \ldots, V_n \) be the sets of nodes that are connected to \( v \) by its \( r \)-hyperedges. \( V_1, \ldots, V_n \) are all distinct because a ground hypergraph cannot have more than one \( r \)-hyperedge connected to a set of nodes. (An \( r \)-hyperedge corresponds to a true ground atom, and each true ground atom can only appear once in a database.) Note that \( p^n_s = p^n_s r_1 V_1 r_1 V_1 \ldots r_n V_n r^n_n \) is a path from \( s \) to \( v \). We cannot create a path \( p^{n'}_s \) that is symmetrical to \( p \) because \( p^{n'}_s \) can contain at most \( n' < n \) distinct set of nodes that are connected by \( r \)-hyperedges to \( v' \). Hence we arrive at a contradiction that \( v \) and \( v' \) are not symmetrical. \( \square \)

**Proposition 1** Let \( v, v' \) and \( s \) be nodes in a ground hypergraph whose nodes are all reachable from \( s \), and \( \text{Sym}_s(v,v') \). If an \( r \)-hyperedge connects \( v \) to a node set \( W \), then an \( r \)-hyperedge connects \( v' \) to a node set \( W' \) that is symmetrical to \( W \).

**Proof.** Suppose for a contradiction that \( v' \) is not connected by any \( r \)-hyperedge to a node set that is symmetrical to \( W \). Let \( v \) and \( v' \) each be respectively connected by \( n \) \( r \)-hyperedges (by Lemma 1) to node sets \( W_1, \ldots, W_n \) and \( W'_1, \ldots, W'_n \) where \( n \geq 1 \) and \( W_1 = W \). \( \pi_i \) denotes a path from \( s \) to \( W_i \) to \( v \) via \( r \), and then back to \( s \) via the reverse path, i.e., \( \pi_i = p^W_s r v p^W_s \). Let \( \Pi_i = \{ \pi_i \} \) be the set of all such paths. Similarly \( \pi'_i = p^W_s r v' p^W_s \), and \( \Pi'_i = \{ \pi'_i \} \). Let \( Q = \{ \pi_1 \pi_2 \ldots \pi_n \} \) be the set of paths formed by concatenating \( \pi_i \in \Pi_i \). Finally let \( Q = \{ q^1 q^2 \ldots q^m p^r_s \} (m = 1, \ldots, \infty) \) be the set of paths formed by concatenating \( q^j \in Q \), followed by a path from \( s \) to \( v \). Since \( v' \) is symmetrical to \( v \), there exists \( Q' = \{ q^j p^r_s \ldots q^m p^r_s \} \) where \( q^j \in Q' \) such that \( Q' \) is symmetrical to \( Q \). Observe that the \( p^W_s \) prefix of each path in \( Q \) corresponds to the \( p^W_s \) prefix of each path in \( Q' \). Since \( W_1 \) and \( W'_1 \) are not symmetrical, there is a path in \( Q \) that cannot be bijectively mapped to \( Q' \) (or vice versa). Hence \( v \) and \( v' \) are not symmetrical, which contradicts the assumption that they are. \( \square \)

**Proposition 2** The maximum value of \( L_{W,C}(X) \) is attained at \( W = W_0 \) and \( C = C_0 \) where \( C_0 \) is the set of all possible conjunctions of positive ground literals that are true in \( X \), and \( W_0 \) is the set containing the globally optimal weights of the conjunctions.

**Proof.** Suppose for a contradiction \( L_{W_1,C_1}(X) > L_{W_0,C_0}(X) \). First consider \( W_1 \neq W_0, C_1 = C_0 \). This case is not possible because by definition \( W_0 \) are the optimal weights for \( C_0 \). Next consider \( C_1 \neq C_0 \). Each conjunction in \( C_1 \) add all its groundings to a new set \( C_2 \). Each ground conjunction in \( C_2 \) inherits the weight of the conjunction from which it is formed. (If \( C_1 \) contains ground conjunctions, then \( C_1 = C_2 \).) If \( C_2 \) contains fewer conjunctions than \( C_0 \), add these missing ground conjunctions to \( C_2 \) and give them zero weights. \( (W_2, C_2) \) thus created is equivalent to \( (W_1, C_1) \), and hence \( L_{W_2,C_2}(X) > L_{W_0,C_0}(X) \). Since \( C_2 = C_0 \), we contradict the assumption that \( W_0 \) contains optimal weights. \( \square \)